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On the nonlinear stability of Maxwellian states for discrete velocity models of the extended Boltzmann equation

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Abstract. By introducing appropriate weights in the energy method we propose a new technique for studying the nonlinear stability of Maxwellian states for discrete velocity models (McKean and Broadwell) of the extended Boltzmann equation.

1. Introduction

Recently discrete velocity models of the Boltzmann equation have become quite popular, both for their fluid dynamics applications [1] and for the mathematical problems they have raised [2]. Among the latter, a significant role is played by the stability of equilibria (the so-called Maxwellian states) and related relaxation problems [3, 4]. In some recent papers [5, 6] such a question has been addressed for the extended version of the simplest and most ancient discrete models (Carleman, McKean and Broadwell). In such a context, test particles are allowed to interact with the field particles of a given background by absorption and generation events, in addition to the usual elastic scattering process. External sources may also be present. This new feature adds simply linear and constant terms to the usual quadratic terms in the collision part of the kinetic equations; however, this changes considerably the structure of the equations themselves. In [5, 6] the linear stability of Maxwellian states to space-dependent perturbations was studied. In this paper we shall perform nonlinear stability analysis on the real line in the presence of sources. This requires a generalization of the standard energy methods, which have always proved very powerful and effective in fluid dynamics [7], but fail for the considered kinetic problems. If the generalization may be guessed properly by clever inspection for the Carleman model, a specific careful treatment is needed for the other less elementary models.

In section 2 the modified energy method is presented and discussed, with reference to a set of n semilinear hyperbolic partial differential equations with quadratic nonlinearities describing a general discrete velocity model in kinetic theory, even though more general nonlinear terms could be easily included. The established differential inequality determines a criterion for nonlinear stability, and guarantees exponential relaxation to equilibrium when the initial perturbation is small enough. The criterion can be checked analytically for $n = 2$, and may then be applied in order to prove stability for all two-velocity models. In higher dimensions ($n \geqslant 3$) a complete analytical solution of the related minimum problem is not practicable, in general, but significant information may be obtained from the analysis.

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Section 3 is devoted to the extended McKean two-velocity model ($n = 2$). Multiple Maxwellian states may occur in this case, and the stability of all of them for varying parameters is studied analytically. In the final section the extended Broadwell model in three dimensions is illustrated as a prototype of the stability problems to be dealt with when $n \geqslant 3$. Even for the values of parameters excluded by analytical results, numerical verification indicates actual stability in all of the many cases tested.

2. Modified energy method

Suppose that $u(x, t) \in R^n$ satisfies

$$u_t + Du_x = Au + F(u) \tag{1}$$

where D is a diagonal $n \times n$ real matrix, A is an $n \times n$ real matrix and $F(u) \in R^n$ is given by

$$F(u)_i = \sum_{j,k} f_{ijk} u_j u_k \qquad i = 1, \ldots, n \tag{2}$$

where f_{ijk} are some real numbers. We assume that $u(\cdot, t) \in L^2(R)^n$ and $u_x(\cdot, t) \in L^2(R)^n$ for $t \geqslant 0$, that is, $u(\cdot, t) \in H^1(R)^n$ for $t \geqslant 0$. This is the most general form of the extended discrete Boltzmann equation for a single gas in a participating background medium when all interactions are binary, after translating the considered Maxwellian state to the origin.

In [5] and [6] we studied the linearized version of (1) and (2) for some special important models (Carleman, McKean, Broadwell) and concluded their stability. Here we would like to show that solutions of the corresponding nonlinear systems decay exponentially for all choices of parameters, provided that the initial values of u_i are small enough. We use a modified energy method with a careful selection of weights. Our approach is general enough to be applicable, in principle, to any problem of the kind (1) and (2), and thus to essentially any discrete velocity model of the extended Boltzmann equation. A special case was worked out directly in [6] for the extended Carleman model. The McKean and Broadwell models will be specifically considered in sections 3 and 4.

Let W be a diagonal matrix with diagonal entries $w_i \equiv W_{ii} > 0$, $i = 1, \ldots, n$. Taking a scalar product, denoted as usual by $(\cdot, \cdot)$, of (1) with Wu gives

$$(Wu, u)_t + (WDu, u)_x = 2(WAu, u) + 2(WF(u), u). \tag{3}$$

Let λ_w be the smallest of the real numbers λ for which

$$(WAz, z) \leqslant \lambda(Wz, z) \qquad \text{for all } z \in R^n. \tag{4}$$

Thus

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}(Wu, u)\,\mathrm{d}x \leqslant \lambda_w \int_{-\infty}^{\infty}(Wu, u)\,\mathrm{d}x + \int_{-\infty}^{\infty}(WF(u), u)\,\mathrm{d}x. \tag{5}$$

Using Holder's inequality twice gives

$$\begin{aligned}
(WF(u), u) &= \sum_{ijk} w_i f_{ijk} u_i u_j u_k = \sum_{jk}\left(\sum_i w_i f_{ijk} u_i\right) u_j u_k \\
(WF(u), u)^2 &\leqslant \left(\sum_{jk} w_j w_k u_j^2 u_k^2\right) \sum_{jk} \frac{(\sum_i w_i f_{ijk} u_i)^2}{w_j w_k} \\
&= (Wu, u)^2 \sum_{jk} \frac{(\sum_i w_i f_{ijk} u_i)^2}{w_j w_k} \\
&\leqslant C_1^2 (Wu, u)^3 \qquad \text{where } C_1 = \left(\sum_{ijk} \frac{w_i f_{ijk}^2}{w_j w_k}\right)^{1/2}.
\end{aligned} \tag{6}$$

Using (6) in (5) gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}(Wu,u)\,\mathrm{d}x \leqslant \lambda_w \int_{-\infty}^{\infty}(Wu,u)\,\mathrm{d}x + C_1(Wu,u)_{\infty}^{1/2}\int_{-\infty}^{\infty}(Wu,u)\,\mathrm{d}x \tag{7}$$

where $(Wu,u)_\infty \equiv \sup_x(Wu(x,t),u(x,t))$.

Taking a derivative of (1) with respect to x and taking the scalar product with Wu_x gives

$$\begin{aligned}(Wu_x,u_x)_t + (WDu_x,u_x)_x &= 2(WAu_x,u_x) + 2(WF(u)_x,u_x)\\ &\leqslant 2\lambda_w(Wu_x,u_x) + 2(WF(u)_x,u_x).\end{aligned} \tag{8}$$

Note that

$$(F(u)_i)_x = \sum_{jk}(f_{ijk}+f_{ikj})u_k(u_j)_x$$

so hence we can proceed, as in (6), to obtain

$$(WF(u)_x,u_x) \leqslant C_2(Wu,u)^{1/2}(Wu_x,u_x) \qquad \text{where } C_2 = \left(\sum_{ijk}\frac{w_i(f_{ijk}+f_{ikj})^2}{w_j w_k}\right)^{1/2}. \tag{9}$$

Using (9) in (8) and integrating gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}(Wu_x,u_x)\,\mathrm{d}x \leqslant \lambda_w \int_{-\infty}^{\infty}(Wu_x,u_x)\,\mathrm{d}x + C_2(Wu,u)_{\infty}^{1/2}\int_{-\infty}^{\infty}(Wu_x,u_x)\,\mathrm{d}x. \tag{10}$$

Define $E(t) \geqslant 0$ by

$$E(t)^2 = \int_{-\infty}^{\infty}((Wu,u)+(Wu_x,u_x))\,\mathrm{d}x. \tag{11}$$

Adding (7) and (10) gives

$$EE' \leqslant \lambda_w E^2 + C_3(Wu,u)_{\infty}^{1/2}E^2 \qquad C_3 = \max\{C_1, C_2\}. \tag{12}$$

Note that $f(x)^2 = 2\int_{-\infty}^{x} f(t)f'(t)\,\mathrm{d}t \leqslant 2\|f\|_2\|f'\|_2$ and hence

$$\|f\|_\infty^2 \leqslant \|f\|_2^2 + \|f'\|_2^2 \qquad \text{for } f \in H^1(R). \tag{13}$$

(13) implies that $(Wu,u)_\infty^{1/2} \leqslant E$ and therefore

$$E' \leqslant \lambda_w E + C_3 E^2 \tag{14}$$

which implies that *if $\lambda_w < 0$ then $E(t) \to 0$ exponentially as $t \to \infty$ provided that* $E(0) < -\lambda_w/C_3$. In other words, if $\lambda_w < 0$ then all solutions of (1), which are small enough initially, decay exponentially.

2.1. About λ_w

Let A be a $n \times n$ real matrix and let W be a diagonal matrix with diagonal entries $w_i \equiv W_{ii} > 0$, $i = 1, \ldots, n$. We defined λ_w to be the smallest of the real numbers λ for which

$$(WAz,z) \leqslant \lambda(Wz,z) \qquad \text{for all } z \in R^n. \tag{15}$$

Note that λ_w is the largest eigenvalue of

$$A_w \equiv \tfrac{1}{2}(W^{1/2}AW^{-1/2} + W^{-1/2}A^*W^{1/2}). \tag{16}$$

Consider, for example,

$$A = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}. \tag{17}$$

The matrix has eigenvalues -1. If we simply take $w_i = 1$, as in the usual energy approach, then $\lambda_w = 1$, which is useless in (14)—we need $\lambda_w < 0$. In general, $\lambda_w = -1 + 2\sqrt{w_1/w_2}$ in this example.

Thus, for a given A we need to find weights w_i that give the smallest λ_w. If μ is any eigenvalue of A and $Ax = \mu x$ then

$$\operatorname{Re}\mu\,(Wx, x) = (A_w W^{1/2}x, W^{1/2}x) \leqslant \lambda_w (W^{1/2}x, W^{1/2}x) \tag{18}$$

which implies

$$\operatorname{Re}\mu \leqslant \lambda_w. \tag{19}$$

Thus, there exists $\inf_{w_i>0}\lambda_w$ and we will denote it by Λ. Note that Λ may not be attained as the above example shows. In case of 2×2 matrices it is easy to see directly that

$$\Lambda = \frac{A_{11}+A_{22}}{2} + \left(\left(\frac{A_{11}-A_{22}}{2}\right)^2 + \max\{0, A_{12}A_{21}\}\right)^{1/2}. \tag{20}$$

Λ is actually attained when $A_{12}A_{21} \neq 0$, and the corresponding weights are given by $w_1/w_2 = |A_{21}/A_{12}|$. Note that when $A_{12}A_{21} \geqslant 0$ then Λ is equal to the largest eigenvalue of the 2×2 matrix A, whereas Λ is greater than the real part of any eigenvalue of A when $A_{12}A_{21} < 0$. Finding Λ for larger matrices is much more involved.

Note that λ_w is also the largest eigenvalue of

$$\tfrac{1}{2}(A + W^{-1}A^*W). \tag{21}$$

Assume it to be simple and let z be the corresponding eigenvector. Hence

$$(WA + A^*W)z = 2\lambda_w Wz. \tag{22}$$

An infinitesimal change $\mathrm{d}W$ produces changes $\mathrm{d}z$, $\mathrm{d}\lambda_w$ which satisfy

$$(\mathrm{d}W A + A^*\,\mathrm{d}W)z + (WA + A^*W)\,\mathrm{d}z = 2\,\mathrm{d}\lambda_w\, Wz + 2\lambda_w\,\mathrm{d}W\, z + 2\lambda_w W\,\mathrm{d}z. \tag{23}$$

Taking a scalar product with z gives

$$((\mathrm{d}W A + A^*\,\mathrm{d}W)z, z) = 2\,\mathrm{d}\lambda_w (Wz, z) + 2\lambda_w(\mathrm{d}W\, z, z) \tag{24}$$

which simplifies to

$$\mathrm{d}\lambda_w (Wz, z) = (Az - \lambda_w z, \mathrm{d}W\, z) \tag{25}$$

and therefore

$$\frac{\partial\lambda_w}{\partial w_i} = \frac{z_i(Az - \lambda_w z)_i}{(Wz, z)} \qquad i = 1, \ldots, n. \tag{26}$$

When the multiplicity of λ_w is bigger than 1 then (26) gives different directional derivatives that correspond to different eigenvectors and hence the λ_w surface has a corner in such a case.

When the minimum λ_w is attained and is not in a corner then one can use (26) to find it. Equation (26) suggests that we examine various cases separately. In the case when all $z_i \neq 0$ we have to have

$$Az = \lambda_w z \qquad \text{and} \qquad (WA + A^*W)z = 2\lambda_w Wz \tag{27}$$

and hence

$$Az = \lambda_w z \qquad \text{and} \qquad A^* W z = \lambda_w W z. \tag{28}$$

Thus, in this case, the smallest λ_w equals the largest eigenvalue of A and it occurs when

$$w_i = y_i / z_i \qquad \text{for } i = 1, \ldots, n \tag{29}$$

where $y \neq 0$ is such that $A^* y = \lambda_w y$. Thus (29) provides the optimal choice of the weights for all values of the matrix elements in A for which the previous assumptions hold.

3. Extended McKean model

The most typical two-velocity model of the extended Boltzmann equation, together with the Carleman model considered already in [6], is the extended McKean model, given by

$$\frac{\partial N_1}{\partial t} + \frac{\partial N_1}{\partial x} = -N_1 N_2 + N_2^2 - \varepsilon N_1 + \chi \eta \rho + S_1 \tag{30}$$

$$\frac{\partial N_2}{\partial t} - \frac{\partial N_2}{\partial x} = N_1 N_2 - N_2^2 - \varepsilon N_2 + (1-\chi)\eta\rho + S_2 \tag{31}$$

where $\rho = N_1 + N_2$. $\varepsilon > 0$, $\eta > 0$ and $0 \leqslant \chi \leqslant 1$ are constants. $S_i \geqslant 0$ are sources and $S = S_1 + S_2 > 0$. ε represents an absorption coefficient, and η takes generation processes (such as fission for neutrons or ionization for electrons) into account. The factor χ describes the angular distribution of generated particles. The standard McKean model used in classical kinetic theory corresponds to the limiting case $\eta = \epsilon = S = 0$.

Non-trivial, non-negative, constant solutions of (30) and (31) are called Maxwellian states. To obtain them, add (30) to (31), which gives again $\rho(\varepsilon - \eta) = S$ and $\varepsilon > \eta$. Solving (31) when $(1-\chi)\eta\rho + S_2 > 0$ gives

$$N_2 = \frac{\rho - \varepsilon}{4} + \left(\frac{(\rho-\varepsilon)^2}{16} + \frac{(1-\chi)\eta\rho + S_2}{2} \right)^{1/2} \qquad N_1 = \rho - N_2. \tag{32}$$

When $(1-\chi)\eta\rho + S_2 = 0$, i.e. when $\chi = 1$ and $S_2 = 0$, then we have

$$N_2 = 0 \qquad N_1 = \rho \tag{33}$$

and when $\rho > \varepsilon$, i.e. $S > \varepsilon(\varepsilon - \eta)$, we also have another solution

$$N_2 = \frac{\rho - \varepsilon}{2} \qquad N_1 = \frac{\rho + \varepsilon}{2}. \tag{34}$$

Perturbations u_i of a Maxwellian state N_1, N_2 satisfy

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x} = (-N_2 - \varepsilon + \chi\eta)u_1 + (2N_2 - N_1 + \chi\eta)u_2 + u_2^2 - u_1 u_2 \tag{35}$$

$$\frac{\partial u_2}{\partial t} - \frac{\partial u_2}{\partial x} = (N_2 + (1-\chi)\eta)u_1 + (N_1 - 2N_2 - \varepsilon + (1-\chi)\eta)u_2 - u_2^2 + u_1 u_2. \tag{36}$$

We will show now that the perturbations decay exponentially, if they are initially small enough, for all values of the parameters, except when the Maxwellian state is given by (33) with $\rho \geqslant \varepsilon$. So, the Maxwellian state given by (33) is nonlinearly stable for $\rho < \varepsilon$ and it loses its stability to the coexisting Maxwellian state (34) when $\rho > \varepsilon$. The Maxwellian state given by (32) is always nonlinearly stable.

Let us examine first the case when the Maxwellian state is given by (33). In this case

$$A = \begin{pmatrix} \eta - \varepsilon & \eta - \rho \\ 0 & \rho - \varepsilon \end{pmatrix}. \tag{37}$$

Equation (20) implies that $\Lambda = \max\{\eta - \varepsilon, \rho - \varepsilon\}$. Thus $\Lambda < 0$ iff $\rho < \varepsilon$, which shows nonlinear stability of the Maxwellian state (33) for $\rho < \varepsilon$.

Let us examine next the Maxwellian states (32) and (34). Note that $N_2 > 0$ and

$$A = \begin{pmatrix} -N_2 - \varepsilon + \chi\eta & 2N_2 - N_1 + \chi\eta \\ N_2 + (1-\chi)\eta & N_1 - 2N_2 - \varepsilon + (1-\chi)\eta \end{pmatrix} \tag{38}$$

and $A_{21} > 0$.

Let us consider first the case $A_{12} \geqslant 0$. Note that Λ, given by (20), is in this case the largest eigenvalue of A. The characteristic polynomial of A can be written as

$$\lambda^2 + (3N_2 - N_1 + 2\varepsilon - \eta)\lambda + (\varepsilon - \eta)(3N_2 - N_1 + \varepsilon) = 0 \tag{39}$$

with roots $\eta - \varepsilon < 0$ and $N_1 - 3N_2 - \varepsilon$. Since $N_2 > 0$, (31) implies that $N_1 - 3N_2 - \varepsilon < 0$ and hence $\Lambda < 0$.

When $A_{12} < 0$ then (20) implies that $\Lambda = \max\{A_{11}, A_{22}\}$. Since obviously $A_{11} < 0$ we only need to show that $A_{22} < 0$ in order to conclude that $\Lambda < 0$. This is obvious for the Maxwellian state (34) since $A_{22} = (\varepsilon - \rho)/2 < 0$ (recall that $\chi = 1$ in this case). The rest of this section is devoted to showing

$$A_{22} = N_1 - 2N_2 - \varepsilon + (1-\chi)\eta < 0 \tag{40}$$

for the Maxwellian state (32). Equation (40) is equivalent to

$$-\frac{\rho - \varepsilon}{4} + 3\left(\frac{(\rho-\varepsilon)^2}{16} + \frac{(1-\chi)\eta\rho + S_2}{2}\right)^{1/2} - (1-\chi)\eta > 0 \tag{41}$$

which is implied by

$$(\rho - \varepsilon)^2 + (8\rho + \varepsilon)\eta(1-\chi) - 2\eta^2(1-\chi)^2 + 9S_2 > 0. \tag{42}$$

When $\chi = 1$, then $S_2 > 0$ for the Maxwellian state (32) and hence (42) holds. Assume from now on that $\chi < 1$. Equation (42) is implied by

$$F(\eta) \equiv (\rho - \varepsilon)^2 + (8\rho + \varepsilon)\eta(1-\chi) - 2\eta^2(1-\chi)^2 > 0. \tag{43}$$

Since $F(0) \geqslant 0$, the concavity of F implies that it is enough to show that $F(\varepsilon) \geqslant 0$ in order to conclude that $F(\eta) > 0$ for $\eta \in (0, \varepsilon)$. To show that $F(\varepsilon) \geqslant 0$ note that

$$F(\varepsilon) = \rho^2 + (6 - 8\chi)\rho\varepsilon + \mathrm{e}^2\chi(3 - 2\chi) \tag{44}$$

and also

$$F(\varepsilon) = (\rho + (3 - 4\chi)\varepsilon)^2 + 9\varepsilon^2(1-\chi)(2\chi - 1). \tag{45}$$

So, when $\chi \geqslant 1/2$, then $F(\varepsilon) \geqslant 0$ by (45). When $\chi < 3/4$, the minimum, with respect to ρ, of (44) occurs at a negative value; however, (44) is non-negative when $\rho = 0$.

4. Broadwell model

Another typical discrete velocity model of the extended Boltzmann equation is the following six-velocity Broadwell model

$$\frac{\partial N_1}{\partial t} + \frac{\partial N_1}{\partial x} = N_3^2 - N_1N_2 - \varepsilon N_1 + \chi_1\eta\rho + S_1 \tag{46}$$

$$\frac{\partial N_2}{\partial t} - \frac{\partial N_2}{\partial x} = N_3^2 - N_1N_2 - \varepsilon N_2 + \chi_2\eta\rho + S_2 \tag{47}$$

$$\frac{\partial N_3}{\partial t} = -\frac{1}{2}(N_3^2 - N_1N_2) - \varepsilon N_3 + \chi_3\eta\rho + S_3 \tag{48}$$

on the line ($-\infty < x < \infty$), where $\rho = N_1 + N_2 + 4N_3$. Parameters $\varepsilon > 0$, $\eta > 0$, $\chi_1 \geqslant 0$, $\chi_2 \geqslant 0$, $\chi_3 \geqslant 0$ are constants such that $\chi_1 + \chi_2 + 4\chi_3 = 1$. $S_i \geqslant 0$ are sources and $S = S_1 + S_2 + 4S_3 > 0$. The physical meaning of all of them is the same as before, with χ_i describing the angular distribution of generated particles.

Maxwellian states are constants N_1, N_2 and N_3 that satisfy (46)–(48) and which are physically relevant, i.e. $N_i \geqslant 0$ and $\rho > 0$. Adding (46)–(48) gives, in this case, $\rho(\varepsilon - \eta) = S$. Therefore,

$$\eta < \varepsilon \qquad \text{and} \qquad \rho = S/(\varepsilon - \eta). \tag{49}$$

This enables us to define the effective $\tilde{\chi}_i$ by

$$\tilde{\chi}_i \varepsilon \rho = \chi_i \eta \rho + S_i \qquad \text{for } i = 1, 2, 3. \tag{50}$$

Note that $\tilde{\chi}_1 + \tilde{\chi}_2 + 4\tilde{\chi}_3 = 1$. Equations (46)–(48) imply, in this case, that

$$N_1 = (q + \tilde{\chi}_1)\rho \qquad N_2 = (q + \tilde{\chi}_2)\rho \qquad N_3 = (-q/2 + \tilde{\chi}_3)\rho \tag{51}$$

where $q = (N_3^2 - N_1 N_2)/(\varepsilon\rho)$ and hence q has to satisfy

$$3q^2/4 + q(\tilde{\chi}_1 + \tilde{\chi}_2 + \tilde{\chi}_3 + \varepsilon/\rho^*) + \tilde{\chi}_1\tilde{\chi}_2 - \tilde{\chi}_3^2 = 0 \tag{52}$$

which implies that

$$q = \frac{2(\tilde{\chi}_3^2 - \tilde{\chi}_1\tilde{\chi}_2)}{\tilde{\chi}_1 + \tilde{\chi}_2 + \tilde{\chi}_3 + \varepsilon/\rho + ((\tilde{\chi}_1 + \tilde{\chi}_2 + \tilde{\chi}_3 + \varepsilon/\rho)^2 + 3(\tilde{\chi}_3^2 - \tilde{\chi}_1\tilde{\chi}_2))^{1/2}}. \tag{53}$$

Let N_1, N_2 and N_3 be a Maxwellian state and assume that perturbations $N_j + u_j(x, t)$ also satisfy (46)–(48). This implies that

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x} = (\chi_1\eta - \varepsilon - N_2)u_1 + (\chi_1\eta - N_1)u_2 + (4\chi_1\eta + 2N_3)u_3 + u_3^2 - u_1u_2 \tag{54}$$

$$\frac{\partial u_2}{\partial t} - \frac{\partial u_2}{\partial x} = (\chi_2\eta - N_2)u_1 + (\chi_2\eta - \varepsilon - N_1)u_2 + (4\chi_2\eta + 2N_3)u_3 + u_3^2 - u_1u_2 \tag{55}$$

$$\frac{\partial u_3}{\partial t} = \left(\chi_3\eta + \frac{N_2}{2}\right)u_1 + \left(\chi_3\eta + \frac{N_1}{2}\right)u_2 + (4\chi_3\eta - \varepsilon - N_3)u_3 - \frac{u_3^2 - u_1u_2}{2}. \tag{56}$$

This case belongs of course to the class defined by (1) and (2), but an analytical determination of Λ is impossible. However, under the conditions for which (29) is valid, the optimal weights are given by (29) itself, and a long calculation gives

$$w_1 = \frac{1}{2N_1(\chi_1 - \chi_2) + N_3(1 - \chi_2 + \chi_1) + 2\chi_1\eta} \tag{57}$$

$$w_2 = \frac{1}{2N_2(\chi_2 - \chi_1) + N_3(1 - \chi_1 + \chi_2) + 2\chi_2\eta} \tag{58}$$

$$w_3 = \frac{4}{N_1\chi_2 + N_2\chi_1 + 2\chi_3(N_1 + N_2 + \eta)} \tag{59}$$

and $\lambda_w = \eta - \varepsilon < 0$—exactly what we need and expect physically. However, note that w_1 and w_2 can become negative when $|\chi_1 - \chi_2|$ is large—so other cases have to be considered. There are six other cases, corresponding to various $z_i = 0$, that have to be considered in the case of 3×3 matrices, and in all of them the proof of existence of a negative λ_w would be required for stability.

We conclude our investigation on the Broadwell model by presenting the results of extensive numerical computations aimed at checking analytical predictions and at gaining more insight into the cases in which such predictions are still missing. More then 400 000 random selections of parameters $0 < \chi_1 < 1$, $0 < \chi_2 < 1 - \chi_1$, $0 < \mathrm{e} < 10$, $0 < \eta < \varepsilon$,

$0 < S_i < 1$ were made and in each case the smallest λ_w was found by checking all seven cases as well as the possibility of corners. *In all cases the smallest λ_w was negative—which indicates the stability of the Maxwellian states for each specific case.*

In about 77% of cases it turned out that the smallest λ_w is obtained by weights given in (57)–(59); for example,

$$\chi_1 = 0.7 \qquad \chi_2 = 0.2 \qquad \varepsilon = 2 \qquad \eta = 1 \qquad S_1 = S_2 = S_3 = 0.6 \\ \Lambda = -1. \tag{60}$$

In about 8% of cases the smallest λ_w is obtained by setting (26) to zero with $z_1 = 0$; for example,

$$\chi_1 = 0.1 \qquad \chi_2 = 0.8 \qquad \varepsilon = 2 \qquad \eta = 1 \qquad S_1 = S_2 = S_3 = 0.6 \\ \Lambda = -0.968\,360. \tag{61}$$

In about 8% of cases the smallest λ_w is obtained by setting (26) to zero with $z_2 = 0$; for example,

$$\chi_1 = 0.8 \qquad \chi_2 = 0.1 \qquad \varepsilon = 2 \qquad \eta = 1 \qquad S_1 = S_2 = S_3 = 0.6 \\ \Lambda = -0.968\,360. \tag{62}$$

In about 7% of cases the smallest λ_w is obtained in a corner—when the multiplicity of λ_w is equal to 2; for example,

$$\chi_1 = 0.7 \qquad \chi_2 = 0.1 \qquad \varepsilon = 2 \qquad \eta = 1 \qquad S_1 = S_2 = S_3 = 0.6 \\ \Lambda = -0.992\,814. \tag{63}$$

These are the only cases that showed up. Choosing $w_i = 1$, as in the usual energy method, gives positive λ_w in each of the above examples (60)–(63).

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